Para-Generalization of Peierls Bracket Quantization

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Abstract

A convenient formalism is developed to treat classical dynamical systems involving ($p = 2$) parafermionic and parabosonic dynamical variables. This is achieved via the introduction of a parabracket which summarizes the paracommutation relations of the corresponding Green components in a unified manner. Furthermore, it is shown that Peierls quantization scheme may be applied to such systems provided that one uses the above mentioned parabracket to express the quantum paracommutation relations. Application of the Peierls scheme also provides the form of the parafermionic and parabosonic kinetic terms in the Lagrangian.

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1 Introduction

Recently, *Parasupersymmetry* [1, 2, 3] and *Fractional Supersymmetry* [4] have been attracting much attention. This may be best explained by noting the achievements of workers in *Supersymmetry* [5]. Afterall parasupersymmetry and fractional supersymmetry may be viewed as generalizations of the ordinary supersymmetry. This is most easily seen in the structure of the defining algebraic expressions.

For the case of $(p = 2)$-parasupersymmetry, it is shown, in the most general setting, that the degeneracy structure is almost fully determined by the defining parasuperalgebra [3]. In fact, for a large class of $(p = 2)$-parasupersymmetric quantum systems one can even define the analog of the Witten index [6] of supersymmetry [3]. Like the Witten index, this integer is a topological invariant linked to the indices of Fredholm (resp. elliptic) operators for the known cases [7]. Physically, it signifies the exactness or breaking of parasupersymmetry [3].

These indications of the similarity between supersymmetry and parasupersymmetry urges one to seek for a better understanding of both the classical and quantum versions of parasupersymmtry. Parasupersymmetric quantum mechanics (PSQM) has been studied to some extent in the framework of specific quantum mechanical examples [1, 2, 8]. Its classical counterpart, however, has not been studied properly, to the best of author's knowledge. A discouraging factor in such a study would be the complicated algebraic struc-

ture of the associated *para-Grassmann variables*. The latter were introduced in the study of *parastatistics* [9] which is directly related with parasupersymmetry.

In 1953, Green [10] proposed a generalization of quantum field theory that allowed for dynamical fields with generalized statistics or parastatistics. Such theories were studied in a series of articles in 60's and 70's before the advent of supersymmetry [11, 9]. Parastatistics of Green has found some application in string theory [12] and provided an alternative point of view for theories with internal symmetries [13]. A thorough reveiw of the subject is provided in Ref. [9].

To relate the new $(p = 2)$-PSQM with the old parastatistics of Green, one may begin with a study of its classical analog. The corresponding classical parasupersymmetric systems will involve para-Grassmann variables ψ of order 2, i.e., $\psi^3 = 0$. As is the case for ordinary fermionic variables $(\psi^2 = 0)$, the Lagrangian formulation is most convenient to study such systems. This observation stems from the fact that fermionic coordinates, due to the form of their kinetic term in the Lagrangian, are proportional with their corresponding conjugate momenta. Thus these systems are indeed constrained and the proper treatment of the constraints is necessary in their (Hamiltonian) canonical quantization [14]. The Lagrangian formulation lacks the apparent difficulties with these first class constraints. A quantization scheme applicable in the framework of Lagrangian mechanics was proposed by Peierls [15]

for bosonic systems, and generalized for fermionic and superclassical systems by De Witt [16]. For a demonstration of the application of this method to supersymmetric systems see Refs. [16, 17].

The aim of the present article is to provide a simple formalism which would allow for a concise and unified treatment of both parafermionic and parabosonic dynamical variables of order 2. This allows for a development of the Lagrangian formulation of para-classical mechanics and a generalization of Peierls quantization scheme. In section 2 a brief reveiw of the algebra of creation and annihilation operators for parafermionic and parabosonic degrees of freedom and their classical counterparts is provided. Section 3 specializes to the case $p = 2$. Here a parabracket is introduced which summarizes the algebra of Green's components and unifies the treatment of both types of degrees of freedom. Section 4 first discusses the Peierls bracket quantization scheme for classical systems involving ordinary fermionic and bosonic variables. A generaliztion of this approach for $(p = 2)$ Green's components is then proposed. In section 5, the Peierls bracket quantization is applied to a simple one-dimensional parafermi system. The requirement of the consistency of the canonical and Peierls quantization methods leads to the determination of the kinetic term in the Lagrangian. Section 6 includes author's final remarks.

2 Green's Parastatistics

As defined in Ref. [13], parafermionic (parabosonic) statistics of order p, is a type of statistics – called *parastatistics* – which restricts the number of identical particles in a totally symmetric (resp. antisymmetric) state to be at most p. Clearly, for $p = 1$, one recovers the ordinary fermionic or Fermi-Dirac (bosonic or Bose-Einstein) statistics.

Parastatistics is generally signified with the following set of algebraic relations [10, 11, 9]:

$$\begin{aligned}
[a_k, [a_l^\dagger, a_m]_\mp] &= 2\delta_{kl}a_m \ , \\
[a_k, [a_l^\dagger, a_m^\dagger]_\mp] &= 2\delta_{kl}a_m^\dagger \mp 2\delta_{km}a_l^\dagger \ , \\
[a_k, [a_l, a_m]_\mp] &= 0 \ ,
\end{aligned} \tag{1}$$

where $a_k^\dagger$ and a_k denote the creation and annihilation operators, $[x, y]_\mp := xy \mp yx$, $\forall x, y$, and the signs $-$ and $+$ correspond to parafermions and parabosons, respectively.

In general, the order p of parastatistics appears as the label of a representation of the algebra a, generated by a_k and $a_k^\dagger$ with the rules (1). An irreducible representation is provided by choosing a unique vacuum state vector $|0\rangle$:

$$a_k|0\rangle = 0 \ , \quad \forall k \ ,$$

and constructing a Hilbert-Fock space $\mathcal{A}$ using the basic vectors:

$$|k_1, \cdots, k_l\rangle := a_{k_1}^\dagger \cdots a_{k_l}^\dagger |0\rangle \ .$$

One can show [11, 9] that in this representation

$$a_k a_l^\dagger = p \delta_{kl} |0\rangle \ ,$$

for some non-negative integer p.

In his original article, Green [10] proposed another (reducible [11]) representation of the algebra a, which involved bilinear algebraic relations rather than the complicated trilinear relations (1). Green defined the algebra b, generated by the generators ζ_k^α, and $\zeta_k^{\alpha\dagger}$, $\alpha = 0, \cdots, p - 1$, and rules:

$$
\begin{aligned}
\left[\zeta_k^\alpha, \zeta_j^{\alpha\dagger} \right]_\mp &= \delta_{kj} \ , \\
\left[\zeta_k^\alpha, \zeta_j^\alpha \right]_\mp &= 0 \ , \\
\left[\zeta_k^\alpha, \zeta_j^{\beta\dagger} \right]_\pm &= \left[\zeta_k^\alpha, \zeta_j^\beta \right]_\pm = 0 \ , \quad (\alpha \neq \beta) \ .
\end{aligned}
\tag{2}
$$

These relations together with the identification:

$$a_k = \sum_{\alpha=0}^{p-1} \zeta_k^\alpha \ , \tag{3}$$

lead to the defining relation of a, i.e., Eqs. (1).

Choosing the same vacuum state vector, $|0\rangle$, requiring:

$$\zeta_k^\alpha |0\rangle = 0 \qquad \forall \alpha, k \ ,$$

and defining a Hilbert-Fock space $\mathcal{B}$ using the basic vectors:

$$|k_1, \alpha_1; \cdots; k_m, \alpha_m\rangle := \zeta_{k_1}^{\alpha_1\dagger} \cdots \zeta_{k_m}^{\alpha_m\dagger} |0\rangle \ ,$$

one obtains a representation of b. In view of Eq. (3), this also provides a representation for a. This representation is known as the *Green representation*

and ζ_k^α are called the *Green components* of a_k. Note that by construction the spaces $\mathcal{A}$ and $\mathcal{B}$ are in one-to-one correspondence with the polynomial rings generated by $a_k^\dagger$ and $\zeta_k^{\alpha\dagger}$, respectively. Thus according to Eq. (3) $\mathcal{A}$ may be viewed as a subring of $\mathcal{B}$. In fact, the physical space to be considered is $\mathcal{A}$ and not $\mathcal{B}$. The latter is introduced for practical convenience.

Another important point is the possibility of the existence of particles with different types of parastatistics. This is especially the case for parasupersymmetric systems. The treatment of this case leads to the introduction of relative parastatistics [11].

Consider two species of particles a and b, with creation and annihilation operators $a_i^\dagger$, a_i and $b_j^\dagger$, b_j, and orders of parastatistics p_a and p_b, respectively. Then, it can be shown [11] that if $p_a \neq p_b$, the operators $a_i^\dagger$ and a_i either commute or anticommute with $b_j^\dagger$ and b_j. If $p_a = p_b =: p$ then there is a set of trilinear relations between these operators. The latter can be more easily expressed in terms of the corresponding Green's components:

$$a_i = \sum_{\alpha=0}^{p-1} \zeta_i^\alpha \ , \quad b_j = \sum_{\alpha=0}^{p-1} \xi_j^\alpha \ , \tag{4}$$

with $a_i|0\rangle = b_j|0\rangle = \zeta_i^\alpha|0\rangle = \xi_j^\alpha|0\rangle = 0$ and $i = 1, \cdots, n_a$ and $j = 1, \cdots, n_b$, for some positive integers n_a and n_b. The following relations express the relative parastatistics of species of particles a to b:

$$[\zeta_i^\alpha, \xi_j^\alpha]_{-\eta} = [\zeta_i^\alpha, \xi_j^{\alpha\dagger}]_{-\eta} = 0 \ , \tag{5}$$

$$\left[\zeta_i^\alpha, \xi_j^\beta\right]_\eta = \left[\zeta_i^\alpha, \xi_j^{\beta\dagger}\right]_\eta = 0 \ , \quad (\alpha \neq \beta) \ , \tag{6}$$

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where $\eta = \pm$ determines the relative statistics. $\eta = +$ (resp. $\eta = -$) corresponds to the relative parabosonic (resp. parafermionic) statistics.

The relative parastatistics is not determined by physical reasoning, however there is a so called *normal relative parastatistics* [11] that generalizes the known case of $p = 1$. It is described as follows:

I) If $p_a \neq p_b$, then $a_i^\dagger$, a_i and $b_j^\dagger$, b_j anticommute if both particles a and b are parafermions. Otherwise, they commute.

II) If $p_a = p_b = p$, then in Eqs. (5) and (6) $\eta = -$, if both a and b are parafermions. Otherwise, $\eta = +$.

The latter case says that two species of parafermionic particles of the same order have parafermionic relative statistics. Whereas a parabosonic particle has relative parabosonic statistics with respect to both parabosonic and parafermionic particles of the same order. This is a direct generalization of the $p = 1$ case.

We conclude this section with a comment on the classical counterparts of the quantum operators encountered above.

In the spirit of the work of Berezin [18], one defines the classical analogs of $a_i^\dagger$, a_i and $\zeta_i^{\alpha\dagger}$, ζ_i^α as generators of algebras defined by the rules given by (1) and (2) with the right hand side set to zero. Again the formula (3) establishes the relation between these algebras. The generators of the former algebra, i.e., the one defined by setting the right hand side of (1) to zero, with

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the sign $(-)$ chosen in Eqs. (1), are called *para-Grassmann variables of order*
p. There is an alternative definition of para-Grassmann variables advocated
by Fillipov et al. [19] which is relevant to fractional supersymmetry. The
latter will not be employed in this article.

3 The $(p = 2)$ Case and the Parabracket

For $p = 2$, the defining relations (1) simplify considerably [10, 11]. One has:

$$
\begin{aligned}
a_k a_l^\dagger a_m \pm a_m a_l^\dagger a_k &= 2\delta_{kl} a_m \pm 2\delta_{lm} a_k \ , \\
a_k a_l a_m^\dagger \pm a_m^\dagger a_l a_k &= 2\delta_{lm} a_k \ , \\
a_k a_l a_m \pm a_m a_l a_k &= 0 \ ,
\end{aligned}
\tag{7}
$$

which can be easily checked using the Green representation (3). In these
equations, the signs $(+)$ and $(-)$ correspond to $(p = 2)$ parafermionic and
parabosonic operators, respectively. Since we would like to treat both of
these operators simultaneously, the introduction of a grading index $\mu = 0, 1$
is convenient, i.e., we attach μ to operators a_k and their Green components
ζ_k^α as a superindex, and interpret a_k^μ and $\zeta_k^{\alpha\mu}$ as parabosonic if $\mu = 0$ and
parafermionic if $\mu = 1$. Now, we can use μ to express the $(\pm)$ signs in
Eqs. (7). In terms of the Green components:

$$
a_i^\mu = \sum_{\alpha=0}^{1} \zeta_i^{\alpha\mu} = \zeta_i^{0\mu} + \zeta_i^{1\mu} \ ,
\tag{8}
$$

one has:

$$
[\zeta_i^{\alpha\mu}, \zeta_j^{\alpha\mu\dagger}]_{(-1)^{\mu+1}} = \delta_{ij} \ ,
$$

$$[\zeta_i^{\alpha\mu}, \zeta_j^{\alpha\mu}]_{(-1)^{\mu+1}} = 0 \,, \tag{9}$$

$$[\zeta_i^{\alpha\mu}, \zeta_j^{\beta\mu\dagger}]_{(-1)^\mu} = [\zeta_i^{\alpha\mu}, \zeta_j^{\beta\mu}]_{(-1)^\mu} = 0 \,, \quad (\alpha \neq \beta) \,. $$

Note that throughtout the rest of this article the Green indices, $\alpha, \beta, \cdots$, take values 0 and 1, for $p = 2$.

It turns out that it is easier to work with self-adjoint ("real") operators (variables). Thus we introduce yet another index $m = 1, 2$ and consider the self-adjoint operators:

$$\theta_{i1}^{\alpha\mu} := \sqrt{\frac{\hbar}{2}}(\zeta_i^{\alpha\mu} + \zeta_i^{\alpha\mu\dagger}) \,, \quad \theta_{i2}^{\alpha\mu} := -i\sqrt{\frac{\hbar}{2}}(\zeta_i^{\alpha\mu} - \zeta_i^{\alpha\mu\dagger}). \tag{10}$$

Now, if one defines the *parabracket* by:

$$[[\theta_{im}^{\alpha\mu}, \theta_{jn}^{\beta\nu}]] := \theta_{im}^{\alpha\mu}\theta_{jn}^{\beta\nu} - (-1)^{\mu\nu+\alpha+\beta}\theta_{jn}^{\beta\nu}\theta_{im}^{\alpha\mu} \,, \tag{11}$$

then the relation:

$$[[\theta_{im}^{\alpha\mu}, \theta_{jn}^{\beta\nu}]] = \hbar\delta_{ij}\delta^{\alpha\beta}\left[i(1-\mu)(1-\nu)\epsilon_{mn} + \mu\nu\delta_{mn}\right] \,, \tag{12}$$

not only summarizes the defining relations (2) and hence (1), but it also includes the statement of the normal relative parastatistics. In Eq. (12), δ and ϵ are the Kronecker delta function and the Levi Civita symbol, respectively.

One might view Eq. (12) as the statement of canonical quantization for the $(p = 2)$ para-classical systems. In fact, the factor $\hbar$ has been introduced so that (12) yields the definition of the classical counterparts of the quantum operators, i.e., $(p = 2)$ parafermionic and parabosonic variables, in the limit $\hbar \to 0$.

The definition of the parabracket (11) may be extended to polynomials in $\theta_{im}^{\alpha\mu}$. This is done by defining it for the monomials, e.g.

$$M := \theta_{i_1 m_1}^{\alpha_1 \mu_1} \cdots \theta_{i_r m_r}^{\alpha_r \mu_r} \ , \quad N := \theta_{j_1 n_1}^{\beta_1 \nu_1} \cdots \theta_{j_s n_s}^{\beta_s \nu_s} \ , \tag{13}$$

by

$$[[M, N]] := MN - (-1)^{\eta(M,N)} NM \ , \tag{14}$$

$$\eta(M, N) := (\sum_{k=1}^{r} \mu_k)(\sum_{l=1}^{s} \nu_l) + r\sum_{l=1}^{s} \beta_l + s\sum_{k=1}^{r} \alpha_k \ , \tag{15}$$

and requiring bilinearity. In the classical limit, for any two polynomials P and Q in $\theta_{im}^{\alpha\mu}$, one has

$$[[P, Q]] = 0 \ . \tag{16}$$

A more substantial result is a generalization of the Jacobi identity. The following lemma can be easily proved by the application of Eqs. (14) and (15).

Lemma 1: *Let M, N, and O be monomials in $\theta_{im}^{\alpha\mu}$ and the parabracket $[[\ ,\ ]]$, is defined by Eq. (14), then the relation:*

$$(-1)^{\eta(M,O)}[[M, [[N, O]]\,]] \ + \ (-1)^{\eta(O,N)}[[O, [[M, N]]\,]] + \tag{17}$$
$$(-1)^{\eta(N,M)}[[N, [[O, M]]\,]] \ = \ 0 \ ,$$

holds as an identity.[1]

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[1] Eq. (17) is a generalization of the the super-Jacobi identity [16] encountered in the study of supersymmetry. Thus it might be called the *para-Jacobi identity.*

Before proceeding further, we would like to make a further remark about Eq. (12). This equation also provides a description of the $(p = 1)$ case. This is done by making the Green indices vanish, i.e., $\alpha, \beta, \cdots = 0$. This reveals the well-known fact that for the bosonic case ($\mu = \nu = 0$), the variables θ_{i2}^{00} correspond to the momenta conjugate to the coordinates θ_{i1}^{00}. This suggests a similar pattern for the $(p = 2)$ case. That is, the parabosonic coordinate variables in the Lagrangian formulation are $\theta_{i1}^{\alpha 0}$. Whereas there is no such restriction on the parafermionic variables. To demonstrate this in a unified notation, one may introduce a collective index $I = (i; m)$, i.e., consider $\theta_I^{\alpha \mu}$, and require that for $\mu = 0$, $I = (i = 1, \cdots, n_{\pi b}; m = 1)$ and for $\mu = 1$, $I = (i = 1, \cdots, n_{\pi f}; m = 1, 2)$, where $n_{\pi b}$ and $2n_{\pi f}$ are the number of parabosonic and parafermionic degrees of freedom, respectively. In view of this notation, one rewrites (12) in the classical limit, as follows:

$$[[\theta_I^{\alpha \mu}, \theta_J^{\beta \nu}]] = 0 . \tag{18}$$

One also must emphasize that the physical dynamical coordinate varaibles are:

$$\psi_I^{\mu} := \sum_{\alpha = 0}^{1} \theta_I^{\alpha \mu} , \tag{19}$$

and not the Green components $\theta_I^{\alpha \mu}$ themselves. In other words, it is the algebra (ring) of polynomials $\mathcal{P}$ in ψ_I^{μ} that serves as the space of physical quantities. In view of Eq. (19), $\mathcal{P}$ is a subalgebra (subring) of the algebra (ring) of polynomials $\mathcal{T}$ in $\theta_I^{\alpha \mu}$. $\mathcal{T}$ is used as a larger space in which the calculations are performed. To extract the physical results, one is bound

to project to the subspace $\mathcal{P}$. $\mathcal{P}$ and $\mathcal{T}$ have some important subspaces. These are the even subalgebras $\mathcal{P}_2$ and $\mathcal{T}_2$, and the subalgebras generated by only the parabosonic (parafermionic) variables ψ_I^1 (resp. ψ_I^0) of $\mathcal{P}$, and $\theta_I^{\alpha 1}$ (resp. $\theta_I^{\alpha 0}$) of $\mathcal{T}$. These are denoted by $\mathcal{P}^\mu$ and $\mathcal{T}^\mu$, respectively.

In view of Eq. (16), one finds that for example the monomials in $\mathcal{T}_2$ either commute or anticommute. In fact, if there is an even number of parafermionic factors in an even monomial it commutes with all the even monomials and two even monomials with odd numbers of parafermionic factors anticommute. Furthermore, the even subalgebras of both $\mathcal{T}^\mu$ and hence $\mathcal{P}^\mu$, ($\mu = 0, 1$), are commutative. This is important, because one would ordinarily like to choose "physical" quantities such as a Lagrangian to be a commutative object. This cannot be fully achieved with ($p = 2$) variables in general. However, one might suffice to require that the Lagrangian be chosen as a linear sum of even monomials each consisiting of an even number of parafermionic factors. We shall offer a justification for the latter requirement in Sec. 4.

In order to carry out the program of Lagrangian mechanics, one also needs a differential calculus for the variables ψ's or alternatively for θ's. The latter also is addressed in the earlier work in parastatistics [9]. The results can be best demonstrated using an extension of the definition of parabracket which also applies to "partial derivatives":

$$[[\frac{\overrightarrow{\partial}}{\partial\theta_I^{\alpha\mu}}, \frac{\overrightarrow{\partial}}{\partial\theta_J^{\beta\nu}}]] \;=\; [[\frac{\overleftarrow{\partial}}{\partial\theta_I^{\alpha\mu}}, \frac{\overleftarrow{\partial}}{\partial\theta_J^{\beta\nu}}]] \;=\; 0, \tag{20}$$

$$[[\theta_I^{\alpha\mu}, \frac{\overrightarrow{\partial}}{\partial\theta_J^{\beta\nu}}]] \;\; = \;\; [[\frac{\overleftarrow{\partial}}{\partial\theta_I^{\alpha\mu}}, \theta_J^{\beta\nu}]] \;\; = \;\; \delta_{\mu\nu}\delta_{\alpha\beta}\delta_{IJ}\,, \tag{21}$$

where one defines the left hand sides of the latter equations by replacing θ's in Eq. (11) by either of $\overleftarrow{\partial}/\partial\theta$ or $\overrightarrow{\partial}/\partial\theta$, with the same indices.

Eqs. (21) may be used to obtain a generalized Leibniz rule. One has:

Lemma 2: *Let M and N be monomials in $\mathcal{T}$ as given by Eq. (13), then*

$$\frac{\overrightarrow{\partial}}{\partial\theta}(MN) = (\frac{\overrightarrow{\partial}}{\partial\theta}M)N - (-1)^{\eta(M,N)}(\frac{\overrightarrow{\partial}}{\partial\theta}N)M\,, \tag{22}$$

where $\eta(M,N)$ is defined by Eq. (15), and the indices of θ's are suppressed for simplicity.

A proof of Lemma 2 involves a lengthy two step induction on the orders r and s of the monomials. Here, one makes extensive use of Eqs. (11), (18) and (21). Eq. (22) is of great practical use in performing computations with θ's. A similar result may be proven for $\overleftarrow{\partial}/\partial\theta$.

We conclude this section with a discussion of the *reality* condition. As is the case in the analysis of supernumbers [16], we define a real element of the (complex) algebra $\mathcal{T}$, and similarly $\mathcal{P}$, by introducing a $*$-operation. This is already implicit in the quantum level in the definition of the Hermitian conjugation. Following the $(p=1)$ case [16], we require

$$\left(\lambda\,\theta_{I_1}^{\alpha_1\mu_1}\cdots\theta_{I_r}^{\alpha_r\mu_r}\right)^* := \lambda^*\theta_{I_r}^{\alpha_r\mu_r}\cdots\theta_{I_1}^{\alpha_1\mu_1}\,, \tag{23}$$

and (additive) linearity of $*$-operation. Here λ is a complex number and λ^*

stands for its complex conjugate. A real element of $\mathcal{T}$ (resp. $\mathcal{P}$) is one whose $*$-conjugate equals itself.

4 Peierls Bracket Quantization

A generalization of Peierls bracket quantization to systems involving bosonic (commuting) and fermionic (anticommuting) dynamical variables is carried out in Ref. [16]. Here a brief review is presented.

Consider a (non-relativistic) classical system whose dynamics is described by the action functional:

$$S[\Phi] := \int_\Phi L(\Phi^i(t), \dot{\Phi}^i(t), t)dt \ , \tag{24}$$

where Φ^i are the coordinate variables, $\dot{\Phi}^i$ are their corresponding velocities, $t \in [0, T]$ is the time variable, and $\Phi = (\Phi(t))$ is a path in the configuration space. Following Ref. [16], let us denote the right and left functional derivatives by

$$S_{,i'} \equiv S[\Phi]\frac{\overleftarrow{\delta}}{\delta \Phi^i(t')} \ , \quad {}_{i'}S \equiv \frac{\overrightarrow{\delta}}{\delta \Phi^i(t')}S[\Phi] \ , \tag{25}$$

respectively. In this (condensed) notation, the indices represent both the discrete and continuous (time) labels and repeated indices imply summation over the discrete and integration over the continuous labels. In particular, note that the prime on the index i in (25) is associated with the continuous index, t'.

The dynamical equations are given by

$$S_{,i} = 0 \ . \tag{26}$$

The second functional derivatives of the action functional yield the Jacobi operator: $(_iS_{,j'})$. The Green's functions of the latter are defined according to their boundary conditions and the familiar relation:

$$_iS_{,j'} \, G^{j'k''} = -\, _i\delta^{k''} \ , \tag{27}$$

where the repeated index j' is summed and integrated over, and

$$_i\delta^{k''} \equiv \delta_i^k \delta(t - t'') \ .$$

Denoting the advanced and retarded Green's functions by G^+ and G^-, one defines the Peierls bracket of the fields $A = A[\Phi^i]$ and $B = B[\Phi^i]$ according to:

$$(A, B) := A_{,i} \, \tilde{G}^{ij'} \,_{j'}B \ , \tag{28}$$

where the Green's function $\tilde{G}$ is defined by

$$\tilde{G} := G^+ - G^- \ , \tag{29}$$

It is called the *supercommutator function* by De Witt [16]. One also has the useful relation:

$$(\Phi^i, \Phi^{j'}) := \tilde{G}^{ij'} \ . \tag{30}$$

The Peierls quantization scheme invloves the promotion of the classical fields to linear operators acting on a Hilbert space and satisfying the following

(not necessarily equal time) supercommutation relations:

$$[\hat{A}, \hat{B}]_{\text{super}} = i\hbar(\widehat{A, B}) . \tag{31}$$

Here the hats are placed to emphasize that the corresponding quantities are operators. They will be dropped where possible. If A and B have definite parity, then the *superbracket* $[\ ,\ ]_{\text{super}}$ becomes the ordinary commutator if either of A or B is bosonic. Otherwise it becomes the anticommutator. In practice, one usually uses Eq. (31) written for the coordinate variables, i.e.,

$$[\hat{\Phi}^i, \hat{\Phi}^{j'}]_{\text{super}} = i\hbar\widehat{G^{ij'}} , \tag{32}$$

and properties of the Peierls bracket [16] (more conveniently those of the superbracket) to compute the superbracket of other fields.

Employing the parity indices μ, ν, $\cdots$ of Sec. 2, i.e., considering $\Phi^{i\mu}$, with $\mu = 0$ corresponding to bosonic coordinates and $\mu = 1$ to the fermionic coordinates, one has

$$[\hat{\Phi}^{i,\mu}, \hat{\Phi}^{j'\nu}]_{\text{super}} := \hat{\Phi}^{i\mu}\hat{\Phi}^{j'\nu} - (-1)^{\mu\nu}\hat{\Phi}^{j'\nu}\hat{\Phi}^{i\mu} . \tag{33}$$

Ref. [16] uses the same indices to label the coordinates and their parity.

For this procedure to make sense, the Peierls bracket must possess a series of properties. These are essentially the properties of the supercommutator, namely the *supersymmetry* property:

$$(A^\mu, B^\nu) = -(-1)^{\mu\nu}(B^\nu A^\mu) , \tag{34}$$

and *super-Jacobi identity*:

$$(-1)^{\mu\pi}(A^\mu, (B^\nu, C^\pi)) + (-1)^{\pi\nu}(C^\pi, (A^\mu, B^\nu)) + (-1)^{\nu\mu}(B^\nu, (C^\pi, A^\mu)) = 0 ,$$
(35)

where A^μ, B^ν and C^π are functions of Φ^i and have definite parities μ, ν and π, respectively. Note that relations (34) and (35) are quite nontrivial. A proof of Eqs. (34) and (35) uses the symmetries of the Jacobi operator $_{i,}S_{,j'}$, the supercommutator function $\tilde{G}^{ij'}$, and their functional derivatives under the exchange of their indices [16].

In view of the developments presented in the last section, we proceed to generalize the Peierls scheme to systems involving $(p = 2)$ parabosonic and parafermionic variables.[2] In order to pursue in this direction, we consider a Lagrangian L built up of parabosonic and parafermionic variables ψ_I^μ ($\mu = 0, 1$) and the corresponding velocities, $\dot{\psi}_I^\mu$, i.e.,

$$L = L(\psi_I^\mu, \dot{\psi}_I^\mu, t) .$$
(36)

Note that the velocities are considered as independent variables with the same parastatistical properties. In general, we shall consider real Lagrangians which are even polynomials in both parafermionic and parabosonic variables. The latter condition will prove essential in having a consistent quantization scheme. For practical purposes, we then switch to the Green's components $\theta_I^{\alpha\mu}$ and $\dot{\theta}_I^{\alpha\mu}$. Using the calculus developed for Green's components, one can

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[2]Inclusion of ordinary fermionic and bosonic variables to such systems can also be carried out within the framework presented in the present article.

define the notion of functional differentiation, e.g., according to

$$
\begin{aligned}
F_{,i'} &\equiv F[\theta(t)]\frac{\overleftarrow{\delta}}{\delta\theta^i(t')} \\
&:= \left(F[\theta^1(t),\cdots,\theta^i(t+\epsilon\delta(t-t')),\cdots,\theta^d(t)] - F[\theta(t)]\right)\left.\frac{\overleftarrow{\partial}}{\partial\epsilon}\right|_{\epsilon=0} ,
\end{aligned}
$$

where the index i is a collective index representing $(I,\ \alpha,\ \mu)$ and ϵ is a variable with the same parastatistical properties as θ^i. The left functional derivative is defined similarly.

Identifying the coordinate variables Φ^i of the beginning of this section by θ^i, with $i \equiv (I,\alpha,\mu)$, the action functional, the dynamical equations, the Jacobi operator, and its Green's functions are given according to Eqs. (24), (26) and (27), respectively. The *para-generalization* of the Peierls bracket is obtained by Eqs. (28) and (29). The following analog of Eq. (31) then yields the Peierls quantization condition:

$$
[[\hat{A},\hat{B}]] = i\hbar(\widehat{A,B}) \tag{37}
$$

where $[[\ ,\ ]]$ is the parabracket defined by Eq. (14), and A and B are polynomials in $\theta_I^{\alpha\mu}$. In particular, one has:

$$
[[\hat{\theta}^i(t),\hat{\theta}^j(t')]] = i\hbar\widehat{\tilde{G}^{ij'}} \ . \tag{38}
$$

The above procedure would be consistent provided that the para-generalized Peierls bracket satisfies the symmetry properties of the parabracket, namely the *parasupersymmetry* properties:

$$
(M,N) = -(-1)^{\eta(M,N)}(N,M) \ , \tag{39}
$$

and the *para-generalized Jacobi identity*:

$$(-1)^{\eta(M,O)}(M,(N,O))+(-1)^{\eta(O,N)}(O,(M,N))+(-1)^{\eta(N,M)}(N,(O,M)) = 0 ,$$

(40)

Here the function η is the one defined by Eq. (15) and M, N, and O are monomials in θ's.

A proof of Eq. (39) follows from the following symmetry property of *paracommutator function* $\tilde{G}^{ij'}$:

Proposition 1: *Let $\tilde{G}^{ij'}$ be defined by Eq. (29), $i = (I,\alpha,\mu)$ and $j = (J,\beta,\nu)$, then:*

$$\tilde{G}^{ij'} = -(-1)^{\mu\nu+\alpha+\beta}\tilde{G}^{j'i} .$$

(41)

To arrive at a proof of Prop. 1, we first state a couple of related results which are labeled as Lemmas 3 and 4:

Lemma 3: *Let $M = \theta_{J_1}^{\gamma_1\rho_1}\cdots\theta_{J_D}^{\gamma_D\rho_D}$ be a monomial of order D, then:*

$$\frac{\overrightarrow{\partial}}{\partial\theta_I^{\alpha\mu}}M = (-1)^{\mu+\sum_{a=1}^{D}\eta(\theta_I^{\alpha\mu},\theta_{J_a}^{\gamma_a\rho_a})}(M\frac{\overleftarrow{\partial}}{\partial\theta_I^{\alpha\mu}}) .$$

(42)

A proof of this result is obtained by a direct computation of both sides of Eq. (42) using the result of Lemma 2, i.e., Eq. (22). Next, we have:

Lemma 4: *Let M be as in Lemma 3, and let i and j label (I,α,μ) and (J,β,μ) respectively, then*

$$\frac{\overrightarrow{\partial}}{\partial\theta^i}M\frac{\overleftarrow{\partial}}{\partial\theta^j} = (-1)^{(1+\sum_{a=1}^{D}\rho_a)(\mu+\nu)+\mu\nu+(D+1)(\alpha+\beta)}\frac{\overrightarrow{\partial}}{\partial\theta^j}M\frac{\overleftarrow{\partial}}{\partial\theta^i} .$$

(43)

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In particular, if M is an even monomial in both parabosonic and para

-fermionic variables, then

$$\frac{\overrightarrow{\partial}}{\partial \theta^i} M \frac{\overleftarrow{\partial}}{\partial \theta^j} = (-1)^{\mu\nu+\mu+\nu+\alpha+\beta} \frac{\overrightarrow{\partial}}{\partial \theta^j} M \frac{\overleftarrow{\partial}}{\partial \theta^i} \ . \tag{44}$$

Lemma 4 is a straightforward consequence of Lemma 3. The statement of Lemma 4 generalizes to the case of functional derivatives as well. Namely:

Corollary: *If the action functional S consists of terms which are even monomials in both parabosonic and parafermionic variables, then one has:*

$$_iS_{,j'} = (-1)^{\mu\nu+\mu+\nu+\alpha+\beta} \ _{j'}S_{,i} \ . \tag{45}$$

Eqs. (27) and (45) together with the observation that both $_iS_{,j'}$ and $G^\pm$ are even polynomials, lead to the desired reciprocity relation:

$$G^{\pm ij'} = (-1)^{\mu\nu+\alpha+\beta} G^{\mp j'i} \tag{46}$$

This equation and the definition (29) yield a proof of Prop. 1.

Proof of the para-Jacobi identity (40) follows essentially the same procedure as in the $(p = 1)$ case [16], but the computations are more involved.

In the next section, we consider a simple example of application of Peierls quantization program for a $(p = 2)$–parafermionic system.

5 One-dimensional Parafermi System

Let us denote by ψ a classical $(p = 2)$–parafermionic (para-Grassmann) variable with the Green components $\tau^\alpha := \theta_{I=1}^{\alpha,\mu=1}$. Then the defining relations

(7), in the classical limit, imply

$$\psi^3 = \dot{\psi}^3 = 0 \,, \quad \psi^2\dot{\psi}^2 = \dot{\psi}^2\psi^2 \,,$$
$$\psi\dot{\psi}^2 = -\dot{\psi}^2\psi \,, \quad \psi^2\dot{\psi} = -\dot{\psi}\psi^2 \tag{47}$$

where ψ and $\dot{\psi}$ are treated as independent $(p = 2)$–parafermi variables. Furthermore, one has the realtions:

$$(\psi\dot{\psi})^2 = (\dot{\psi}\psi)^2 = 0 \,, \tag{48}$$

which are most easily verified using the Green representation.

In view of Eqs. (47) and (48), the most general real even polynomial in dynamical variables – upto an unimportant multiplicative constant and additive total time derivatives – has the form:

$$L = \frac{A}{2}\psi^2 + \frac{B}{2}\dot{\psi}^2 + \frac{C}{4}\psi^2\dot{\psi}^2 + \frac{i}{4}(\psi\dot{\psi} - \dot{\psi}\psi) \,. \tag{49}$$

Here A, B and C are real numerical parameters. Eq. (49) serves as the most general possible form for the Lagrangian. In the following we shall make a further demand, namely that the Peierls bracket quantization and the canonical quantization of this system be consistent.

To carry out Peierls' program we first rewrite the Lagrangian (49) in terms of the Green components τ^α and $\dot{\tau}^\alpha$ and compute the Jacobi operator. Here we suffice to state the results:

$$L = \sigma_{\alpha\beta}\left(\frac{A}{2}\tau^\alpha\tau^\beta + \frac{B}{2}\dot{\tau}^\alpha\dot{\tau}^\beta + \frac{C}{4}\sigma_{\gamma\delta}\tau^\alpha\tau^\beta\dot{\tau}^\gamma\dot{\tau}^\delta\right) + \frac{i}{4}\delta_{\alpha\beta}\tau^\alpha\dot{\tau}^\beta \,, \tag{50}$$

$$\beta', S_{,\alpha} = \left\{\left[\sigma_{\alpha\beta}\left(-B + \frac{C}{2}\sigma_{\gamma\delta}\tau^\gamma\tau^\delta\right)\right]\frac{\partial^2}{\partial t^2} + \right. \tag{51}$$

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$$\left[-i\delta_{\alpha\beta} - C([(-1)^{\alpha+\beta}+1]\sigma_{\alpha\gamma}\sigma_{\delta\beta} - \sigma_{\alpha\beta}\sigma_{\gamma\delta})\tau^{\gamma}\dot{\tau}^{\delta}\right]\frac{\partial}{\partial t} +$$

$$\left[A\sigma_{\alpha\beta} - C([\sigma_{\alpha\delta}\sigma_{\gamma\beta} - \frac{1}{2}\sigma_{\alpha\beta}\sigma_{\gamma\delta}]\dot{\tau}^{\gamma}\dot{\tau}^{\delta} + \sigma_{\alpha\delta}\sigma_{\beta\gamma}\tau^{\gamma}\ddot{\tau}^{\delta})\right]\right\}\delta(t-t') ,$$

where σ denotes the Pauli matrix σ_1, i.e.,

$$\sigma_{\alpha\beta} = \begin{cases} 1 & \text{if} \quad \alpha = \beta \\ 0 & \text{if} \quad \alpha \neq \beta. \end{cases} \tag{52}$$

The Green's functions can be computed as power series in $(t-t')$, similarly to the $(p=1)$ case [16, 17]. A simple analysis of the Green's functions, shows that if $B \neq 0$ or $C \neq 0$, then $[[\tau^{\alpha}(t), \tau^{\beta}(t)]] = 0$, which is inconsistent with the result of canonical quantization (12), namely:

$$[[\tau^{\alpha}(t), \tau^{\beta}(t)]] = \hbar\delta^{\alpha\beta} . \tag{53}$$

Setting $B = C = 0$, and carrying out the computation of the Green's functions, one finds:

$$G^{\pm\alpha\beta'} = \left[\mp i\delta^{\alpha\beta} + O(t-t')\right]\Theta[\pm(t-t')] , \tag{54}$$

where Θ is the step function: $\Theta(t) = 1$ if $t > 0$, $\Theta(t) = 0$ if $t < 0$, $\Theta(0) = 1/2$. The latter relation directly leads to Eq. (53) and confirms the consistency of the canonical and Peierls quantization programs. Enforcing $B = C = 0$ in the expression for the Lagriangian (49), one has:

$$L = \frac{i}{4}(\psi\dot{\psi} - \dot{\psi}\psi) + \frac{A}{2}\psi^2 . \tag{55}$$

The first couple of terms in the right hand side of (55) has the same form as the kinetic term for ordinary fermionic systems. We shall refer to these also

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as the kinetic term of the parafermionic system. The last term serves as a potential term which does not have a counterpart in fermionic systems.

A similar analysis shows that for $(p = 2)$–parabosonic systems, choosing the kinetic term to be of the same form as the bosonic kinetic term, one ensures the consistency of the canonical and Peierls quantization schemes.

6 Conclusion

The Lagrangian formulation of classical mechanics is shown to be applicable to systems involving $(p = 2)$ parafermionic and parabosonic variables. The introduction of the parabracket for the Green's components of the $(p = 2)$ dynamical variables facilitates computations considerably. It also allows for a generalization of the Peierls quantization program to such systems.

The internal consistency of the Peierls program requires the Lagrangian to be an even polynomial in both parafermi and parabose variables. The consistency of the results of the canonical and Peierls quantization programs leads to the specification of the form of the parafermionic and parabosonic kinetic terms in the Lagrangian.

The material developed in this article has direct application in the study of systems involving both the parafermi and parabose variables of order $(p = 2)$. Some examples of such systems have been encountered in the context of parasupersymmetric quantum mechanics [2]. There are still quite a few unsettled issues regarding the true meaning of parafermi-parabose (su-

per)symmetry. Some of these issues are addressed in a companion paper [20] using the formalism developed above.

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